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Hamiltonian structure for singular isomonodromy deformation equations

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Abstract. A Hamiltonian structure is constructed explicitly for a class of isomonodromy deformation equations for ordinary differential equations with one irregular singularity. The Hamiltonian turns out to be closely related to a differential form, which is known to be naturally associated to isomonodromy deformation equations. The monodromy data are constants of motion. The functional independence of the formal monodromy exponent and Stokes matrices is demonstrated by an example.

1. Introduction

A vast amount of work has been done in the last couple of years on systems of nonlinear differential equations which arise from isospectral deformations of linear operators. Many physically interesting Hamiltonian systems have been shown to be obtainable from isospectral deformations. As a consequence, they possess a large number of conserved quantities (like spectra and scattering data) which may even be large enough to prove their complete integrability.

The monodromy group of a linear differential equation is the group which is generated by those linear transformations which correspond to analytic continuation of a given fundamental system of solutions along closed paths with fixed initial and end point. It may be non-trivial, if the coefficients of the differential equations are singular at some points (possibly including infinity). Deformations of the coefficients are called isomonodromic if they do not change the monodromy group. They have been known in the mathematical literature for many decades and lead to complicated nonlinear equations for the coefficients, which, however, can be solved, once one is able to solve the subsidiary linear Riemann-Hilbert problem of finding a fundamental system with prescribed monodromy behaviour.

This isomonodromy method is potentially even stronger than isospectral deformations. Its power transpires most impressibly in the monumental work of Sato, Miwa, Jimbo, Ueno and collaborators, summarised in Jimbo *et al* (1979, 1981), Jimbo and Miwa (1981a, b), where, for instance, correlation functions of many field theoretical models are evaluated. By now, the literature on isomonodromy problems (IMPS) is very extensive.

The main problem we should like to deal with in this paper concerns the Hamiltonian formulation of isomonodromy equations. Also, the inverse question of associating isomonodromy problems to Hamiltonian systems may be of interest. To be more precise, we discuss the deformation problem for a 2×2 matrix linear differential equation

$$\partial Y/\partial\zeta = A(\zeta)Y \tag{1.1}$$

under the hypothesis that the coefficients $A(\zeta)$ have only one singularity, which is assumed to be of irregular type and situated at $\zeta=\infty$. The parameters we are going to vary are the exponents T_{-1} of a formal solution of (1.1), which are assumed to be linear functions of a parameter x. The monodromy data which are to be kept fixed include the so-called Stokes matrices S_j, which connect the analytic continuation of a fundamental solution with a certain asymptotic expansion in a sector $\mathscr{S}_j$ with centre at $\zeta=\infty$ to a fundamental solution with the same asymptotic expansion in the neighbouring sector $\mathscr{S}_{j+1}$.

Quite generally, and also in our case, the isomonodromy condition is a compatibility condition between (1.1) and another linear equation

$$(\partial Y/\partial x)\,\mathrm{d}x = \Omega(x)Y \tag{1.2}$$

where x is the deformation parameter, and Ω is a 1-form whose construction in terms of the coefficient A of (1.1) is described e.g. in Jimbo *et al* (1981). The compatibility condition between (1.1) and (1.2) is

$$(\partial A/\partial x)\,\mathrm{d}x - \partial\Omega/\partial\zeta - [\Omega, A] = 0. \tag{1.3}$$

The formulation as a compatibility condition exhibits an important analogy between isomonodromic and isospectral deformations.

Our strategy will be the following. We start out from equation (1.3) rather than (1.1) and assume $\Omega := \bar{\Omega}\,\mathrm{d}x$ to be of the special form

$$\bar{\Omega} = -(U + D\zeta) \tag{1.4}$$

where

$$U = \begin{pmatrix} 0 & u_1(x) \\ u_2(x) & 0 \end{pmatrix} \tag{1.5}$$

and D is a constant traceless diagonal matrix

$$D = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}. \tag{1.6}$$

Equation (1.3) then assumes the form

$$\partial A/\partial x - [U + D\zeta, A] = -D. \tag{1.7}$$

The corresponding linear homogeneous equation

$$\partial A/\partial x - [U + D\zeta, A] = 0 \tag{1.8}$$

has been treated in considerable detail by Dickey (1981). Considered as an equation for A it is shown to have formal solutions

$$A=\sum_{k=0}^{\infty} A_k \zeta^{-k} \tag{1.9}$$

identically in $u_1(x)$ and $u_2(x)$, where the coefficients A_k are in the differential algebra generated by u_1 and u_2. The requirement that the series for A breaks off after the mth term gives a pair of nonlinear differential equations for u_1, u_2, whose solutions yield a family of finite series solutions $A^{(m)}$ of (1.8). Dickey gives Hamiltonian structures for the differential equations which arise from these break-off conditions.

We were able to relate finite series solutions of (1.8) and (1.7) and to find a Lagrangian, a symplectic form and a Hamiltonian for the break-off equations corresponding to (1.7).

This means that we have cast the isomonodromy deformation equations for (1.1) with Ω in the form given above into explicit Hamiltonian form, once we have shown that (1.7) is really the deformation equation for (1.1) if A is taken as the solution of (1.7) with the prescribed Ω. This amounts to showing that the steps leading to the isomonodromy conditions can be reversed.

We shall present two proofs that this can really be achieved. Then, in particular, it can be deduced that the monodromy data are integrals of motion of the Hamiltonian system, which is equivalent to the deformation problem.

Moreover, we demonstrate that a certain function ω, defined in Jimbo *et al* (1981), which can be naturally associated to isomonodromy problems, coincides also for our isomonodromy problem with a multiple of the Hamiltonian.

The Stokes matrices are bound to be integrals of motion, but one might suspect that they are just trivial constants. In order to exclude this dull possibility, we explicitly evaluate the Stokes matrices for the case $m=1$. Normally, the calculation of the Stokes matrices for a given differential equation is a difficult task, and no general procedure for obtaining them is known. In this special case one relates the equation to a case treated by Sibuya in his monograph (1975a). We find that the Stokes matrices are, indeed, not trivial but rather depend on u_1 and u_2 in a perhaps surprisingly complicated way. Their conservation can be explicitly checked.

The material of this paper is organised in the following way. Section 2.1 contains a precise description of the problem and a more detailed outline of our strategy. The construction of the isomonodromy problem is described in §§ 2.2 and 2.3. First in § 2.2 the relation to Dickey's isospectral equation (1.8) is explained and, for completeness, Dickey's results are stated as far as they are relevant for us. Then in § 2.3 the method performing the transition to our inhomogeneous equation (1.7) is explained and the Lagrangian, the symplectic form and the Hamiltonian for the isomonodromy problem are constructed.

In § 3 the isomonodromic deformation is interpreted in terms of the Hamiltonian structure developed in § 2. After some preliminary technical work in § 3.1, in §§ 3.2 and 3.3 we show how to conclude from (1.7) and (1.2) that (1.7) is really the isomonodromy equation for the linear systems (1.1) with the coefficients determined from the Hamiltonian system. Further we show explicitly how the invariance of the monodromy data emerges from the deformation equation. Section 3.4 relates the Hamiltonian of our system to the function ω of Jimbo *et al* (1981). In § 3.5 equations, symplectic form, Lagrangian and Hamiltonian are explicitly written down in an illustrative example. Sections 4.1–4.3 contain the calculation of the Stokes matrices for $m=1$, which are demonstrated to be conserved and non-trivial. The main results, definitions and notation for the isomonodromy problem with a strong singularity at $\zeta=\infty$ are collected in an appendix.

2. Construction of an isomonodromy problem from an appropriate Hamiltonian structure

2.1. The deformation equation as an integrability condition

We want to associate an isomonodromy problem to a Hamiltonian system, such that the Hamiltonian equations of motion become the deformation equations of the IMP and the monodromy data are conserved quantities of the Hamiltonian system. The Hamiltonian function will turn out to be related to the 1-form ω of the IMP.

To be more precise, we choose

$$T_{-1}=\begin{pmatrix} t_{-1,1} & 0 \\ 0 & t_{-1,2} \end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}x+\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \tag{2.1}$$

as the deformation parameter of the singular IMP of the differential equation

$$\partial Y/\partial\zeta=A(\zeta)Y \tag{2.2}$$

with a single strong singularity at $\zeta=\infty$.

In order to construct the Hamiltonian system, we start out from the fact that both the deformation equations of (2.2) and certain integrable Hamiltonian systems arise as integrability conditions of systems of linear partial differential equations.

The isomonodromy deformation equations of the 2×2 matrix differential equation

$$\partial Y/\partial\zeta=A(\zeta)Y, \tag{2.3}$$

with $A(\zeta)$ rational in ζ, are of the general form

$$\mathrm{d}A=\partial\Omega/\partial\zeta+[\Omega, A] \tag{2.4}$$

where d denotes the exterior derivative with respect to the deformation parameters and Ω is a certain 1-form in parameter space, whose construction from the coefficient A can be found in the appendix.

As mentioned above, in our case we have only one deformation parameter, T_{-1}, given by (2.1). Hence $\mathrm{d}=\mathrm{d}x\,\partial/\partial x$. So we have to find a Hamiltonian system with equation of motion

$$\partial A/\partial x=\partial\bar{\Omega}/\partial\zeta+[\bar{\Omega}, A] \tag{2.5}$$

with $\Omega=\bar{\Omega}\,\mathrm{d}x$.

The idea is to invert the procedure which constructs Ω from A and to determine A from an appropriately given matrix Ω.

The equation

$$(\partial A/\partial x)\,\mathrm{d}x-\partial\Omega/\partial\zeta=[\Omega, A] \tag{2.6}$$

is the integrability condition of the system

$$(\partial/\partial x)Y=\bar{\Omega}Y, \tag{2.7}$$

$$(\partial/\partial\zeta)Y=AY. \tag{2.8}$$

We want x to be the time coordinate and ζ to enter only as a parameter in a simple way. This suggests the ansatz

$$\Omega = \hat{\Omega} \equiv -(U+\zeta D)\,\mathrm{d}x \tag{2.9}$$

with

$$U(x) = \begin{pmatrix} 0 & u_1(x) \\ u_2(x) & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}. \tag{2.10}$$

Then equation (2.6) becomes

$$\partial A/\partial x + [U+\zeta D, A] = -D. \tag{2.11}$$

Equation (2.7) now looks like an eigenvalue problem with eigenvalue ζ and a potential given by the functions u_1 and u_2. To find the second equation (2.8) one has to solve equation (2.11) for A.

2.2. *Hamiltonian structure of the homogeneous equation* $(\partial/\partial x)A_0 + [U+\zeta D, A_0] = 0$

To this end we first consider the homogeneous equation

$$(\partial/\partial x)A_0 + [U+\zeta D, A_0] = 0. \tag{2.12}$$

This equation, even for $n \times n$ matrices, has been treated by Dickey (1981); in order to make our presentation reasonably self-sustained and to fix our notations we recollect his results as far as they are relevant for us.

Dickey starts with the isospectral deformation equation

$$R' + [U+\zeta D, R] = 0, \qquad = \partial/\partial x, \tag{2.13}$$

with U and D as in equation (2.10), looking for solutions which are formal series

$$R = \sum_{k=0}^{\infty} R_k \zeta^{-k} \tag{2.14}$$

with coefficients in the differential algebra generated by u_1 and u_2. R is decomposed into

$$R = b_1 R^{(1)} + b_2 R^{(2)} \tag{2.15}$$

where $R^{(1)}$ and $R^{(2)}$ are special solutions of (2.13) with

$$R_0^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R_0^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.16}$$

$R^{(\alpha)}$ $(\alpha = 1, 2)$ is representable as

$$(R^{(\alpha)})_{ij} = \varphi^{i\alpha}\psi^{\alpha j}. \tag{2.17}$$

Here

$$\Phi = \begin{pmatrix} \varphi^{11} & \varphi^{12} \\ \varphi^{21} & \varphi^{22} \end{pmatrix} \tag{2.18}$$

and

$$\Phi^{-1} = \Psi = \begin{pmatrix} \psi^{11} & \psi^{12} \\ \psi^{21} & \psi^{22} \end{pmatrix} \tag{2.19}$$

are formal solutions of the equations

$$\Phi' = (U+\zeta D)\Phi = \Phi\Lambda, \qquad -\Psi' + \psi(U+\zeta D) = \Lambda\Psi,$$
$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{2.20}$$

In the expansions

$$\Phi = \sum_{k=0}^{\infty} \Phi_k \zeta^{-k}, \qquad \Psi = \sum_{k=0}^{\infty} \Psi_k \zeta^{-k}, \qquad \Lambda = \sum_{k=-1}^{\infty} \Lambda_k \zeta^{-k}, \tag{2.21}$$

the coefficients can be determined recursively, and the coefficients of R are finally obtained from Φ_k and Ψ_k.

The quantity

$$\tilde{R}^{(m)} = \sum_{k=0}^{m} R_{m-k} \zeta^k \tag{2.22}$$

which is just given by the first $(m+1)$ terms of the series of $\zeta^m R$ fulfils

$$\tilde{R}^{(m)\prime} + [U+\zeta D, \tilde{R}^{(m)}] = -[D, R_{m+1}] \tag{2.23}$$

iff R solves equation (2.13).

The differential algebra formal series solutions of (2.13) and (2.23) solve the equations identically in the functions u_1 and u_2. Demanding now in addition that the series solution of (2.13) terminates after $(m+1)$ terms leads to the equivalent conditions

$$\tilde{R}^{(m)\prime} + [U+\zeta D, \tilde{R}^{(m)}] = 0 \tag{2.24}$$

or

$$[D, R_{m+1}] = 0 \tag{2.25}$$

which are differential equations for the functions u_1 and u_2.

It is for these equations that Dickey was able to construct a Lagrangian and a completely integrable Hamiltonian system.

A Lagrangian is given by

$$L^{(m)} = \sum_{\alpha=1}^{2} b_\alpha \operatorname{Tr}(DR^{(\alpha)}_{m+2}) \tag{2.26}$$

and accordingly

$$\delta L^{(m)}/\delta U = 0 \tag{2.27}$$

with

$$\frac{\delta}{\delta u_i} = \sum_{k=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^k \frac{\partial}{\partial u_i^{(k)}}, \qquad \frac{\delta}{\delta U} = \begin{pmatrix} \cdot & \delta/\delta u_1 \\ \delta/\delta u_2 & \cdot \end{pmatrix}, \tag{2.28}$$

is equivalent to equation (2.26) or (2.27).

The symplectic form $\mathrm{d}\alpha^{(m)}$ and the Hamiltonian are then obtained from the relations

$$\mathrm{d}L^{(m)} = \sum_{i=1}^{2} \frac{\delta L^{(m)}}{\partial u_i} \mathrm{d}u_i - \frac{\mathrm{d}}{\mathrm{d}x} \alpha^{(m)}, \tag{2.29}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} H^{(m)} = -\sum_{i=1}^{2} u_i' \frac{\delta L^{(m)}}{\delta u_i}. \tag{2.30}$$

The result is

$$H^{(m)} = -(m+1)\,\mathrm{Tr}(DR_{m+2} + UR_{m+1}), \tag{2.31}$$

$$\alpha^{(m)} = -(m+1)\sum_{\alpha=1}^{2} b_\alpha\,\mathrm{Tr}(\varphi^{i\alpha}\,\mathrm{d}\Psi^{\alpha j})|_{m+1}. \tag{2.32}$$

The symbol $|_{m+1}$ means taking the coefficient of ζ^{-m-1}.

From equation (2.24) it is evident that all the eigenvalues of $\tilde{R}^{(m)}$ are conserved quantities, even identically in ζ. Equivalently, $\mathrm{Tr}(\tilde{R}^{(m)})^k$ or the coefficients J_{kl} of the characteristic polynomial

$$\det(\tilde{R}^{(m)} - \omega 1) = \sum_{l=0}^{2} J(\zeta)\omega^l = \sum_{l=0}^{2}\sum_{k=0}^{m(2-l)} J_{kl}\zeta^k\omega^l \tag{2.33}$$

are conserved. The existence of m independent conserved quantities in involution is shown and, moreover, Dickey succeeds in constructing action and angle variables in algebraic terms on a Riemannian surface of genus m.

For us the crucial result is that a Hamiltonian structure for the homogeneous equation (2.12) is found and that the polynomial

$$A_0 = \tilde{R}^{(m)} \tag{2.34}$$

in ζ is a solution of (2.12).

2.3. *Hamiltonian structure of the inhomogeneous equation* $(\partial/\partial x)A + [U + \zeta D, A] = -D$

We now come back to the inhomogeneous equation. We try to solve the equation

$$\partial A/\partial x + [U + \zeta D, A] = -D \tag{2.35}$$

supplemented by a suitable subsidiary condition by adapting Dickey's procedure for the equation

$$\partial A_0/\partial x + [U + \zeta D, A_0] = -[D, R_{m+1}] = 0 \tag{2.36}$$

to this case.

To this end we rewrite equation (2.13) in the form

$$R' + [U, R] = -[\zeta D, R]. \tag{2.37}$$

The recursion relations for the coefficients R_k of the formal expansion are then to be modified in that step which relates R_{m+1} to R_m, because we want to make contact with the inhomogeneous equation. The modified recursion relations for the coefficients $\hat{R}_k$ of a formal power series solution of the inhomogeneous equation are then

$$\begin{array}{llll} (-1) & & 0 & = -[D, \hat{R}_0], \\ (0) & \hat{R}_0' & +[U, \hat{R}_0] & = -[D, \hat{R}_1], \\ & & & \vdots \\ (m-1) & \hat{R}_{m-1}' & +[U, \hat{R}_{m-1}] & = -[D, \hat{R}_m], \\ (m) & \hat{R}_m' + D & +[U, \hat{R}_m] & = -[D, \hat{R}_{m+1}], \\ (m+1) & \hat{R}_{m+1}' & +[U, \hat{R}_{m+1}] & = -[D, \hat{R}_{m+2}]. \end{array} \tag{2.38}$$

The initial condition we choose is

$$\hat{R}_0 = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \qquad \text{i.e. } b_1 = -b_2 = b. \tag{2.39}$$

The first m steps of the recursion are evidently unchanged by the modification and yield $\hat{R}_j = R_j$ for $0 \leqslant j \leqslant m-1$. To find the difference between $\hat{R}_j$ and R_j in the next step, we split into diagonal and off-diagonal parts, denoting the diagonal part of a matrix M by M_{D} and calling $M - M_{\mathrm{D}} = M_{\mathrm{OD}}$. First of all the off-diagonal part of equation $(m-1)$ yields

$$(\hat{R}_m)_{\mathrm{OD}} = (R_m)_{\mathrm{OD}}. \tag{2.40}$$

Then from equation (m):

(i) *Diagonal part*

$$(\hat{R}'_m)_{\mathrm{D}} + D + [U, \hat{R}_m]_{\mathrm{D}} = 0. \tag{2.41}$$

Now, because of $U = U_{\mathrm{OD}}$

$$[U, \hat{R}_m] = [U, (\hat{R}_m)_{\mathrm{OD}}] = [U, (R_m)_{\mathrm{OD}}] = [U, R_m]_{\mathrm{D}}. \tag{2.42}$$

We already know R_m, which fulfils

$$(R'_m)_{\mathrm{D}} + [U, R_m]_{\mathrm{D}} = 0. \tag{2.43}$$

Thus, we get

$$(\hat{R}_m)_{\mathrm{D}} = (R_m)_{\mathrm{D}} - Dx - C. \tag{2.44}$$

Hence, remembering (2.40),

$$\hat{R}_m = R_m - Dx - C. \tag{2.45}$$

For the integration constant C we set

$$C = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}. \tag{2.46}$$

(ii) *Off-diagonal part*

$$(\hat{R}'_m)_{\mathrm{OD}} + [U, \hat{R}_m]_{\mathrm{OD}} = -[D, \hat{R}_{m+1}]. \tag{2.47}$$

From

$$\begin{aligned} [U, \hat{R}_m]_{\mathrm{OD}} &= [U, (\hat{R}_m)_{\mathrm{D}}] \\ &= [U, (R_m)_{\mathrm{D}}] - [U, Dx + C] \\ &= [U, R_m]_{\mathrm{OD}} - [U, Dx + C] \end{aligned} \tag{2.48}$$

it follows that

$$\begin{aligned} (\hat{R}'_m)_{\mathrm{OD}} + [U, \hat{R}_m]_{\mathrm{OD}} &= (R'_m)_{\mathrm{OD}} + [U, R_m]_{\mathrm{OD}} - [U, Dx + C] \\ &= -[D, R_{m+1}] - [U, Dx + C] \\ &= -[D, R_{m+1}] + [D, Ux] + [D, (c/a)U] \\ &= -[D, \hat{R}_{m+1}]. \end{aligned} \tag{2.49}$$

Thus, we have

$$(\hat{R}_{m+1})_{\mathrm{OD}} = (R_{m+1})_{\mathrm{OD}} - (x + c/a)U. \tag{2.50}$$

The quantity

$$\hat{\tilde{R}}^{(m)} := \sum_{k=0}^{m} \hat{R}_{m-k}\zeta^k \tag{2.51}$$

now evidently fulfils

$$\hat{\tilde{R}}^{(m)\prime} + D + [U + \zeta D, \hat{\tilde{R}}^{(m)}] = -[D, \hat{R}_{m+1}]. \tag{2.52}$$

This means that we have constructed a family of matrices

$$A = \hat{\tilde{R}}^{(m)} \tag{2.53}$$

which solve equation (2.35) under the subsidiary condition $[D, \hat{R}_{m+1}] = 0$.

Next, we take over the construction of the Hamiltonian structure to the inhomogeneous case.

Dickey's Lagrangian $L^{(m)} = \mathrm{Tr}\, DR_{m+2}$ yields

$$(\delta L^{(m)}/\delta U)_{\mathrm{OD}} = (m+1)({}^{\mathrm{t}}R_{m+1})_{\mathrm{OD}}, \tag{2.54}$$

where ${}^{\mathrm{t}}M$ means the transposed matrix.

Defining

$$L_+ = -(m+1)(x + c/a)u_1u_2 = (m+1)(x + c/a)\det U \tag{2.55}$$

we obtain

$$\begin{aligned}\left(\frac{\delta L_+}{\delta U}\right)_{\mathrm{OD}} &= \begin{pmatrix} & \delta L_+/\delta u_1 \\ \delta L_+/\delta u_2 & \end{pmatrix} \\ &= \begin{pmatrix} & -(m+1)(x+c/a)u_2 \\ -(m+1)(x+c/a)u_1 & \end{pmatrix} \\ &= -(m+1)(x+c/a)\,{}^{\mathrm{t}}U.\end{aligned} \tag{2.56}$$

Thus putting

$$\hat{L}^{(m)} = L^{(m)} + L_+ = \mathrm{Tr}(DR_{m+2}) - (m+1)(x + c/a)u_1u_2 \tag{2.57}$$

we see that $\delta\hat{L}^{(m)}/\delta U = 0$ is equivalent to $[D, \hat{R}_{m+1}] = 0$. The symplectic form $\mathrm{d}\hat{\alpha}^{(m)}$ is again determined from

$$\frac{\delta\hat{L}^{(m)}}{\delta U} = \sum_{i=1}^{2} \frac{\delta\hat{L}^{(m)}}{\delta u_i}\,\mathrm{d}u_i - \frac{\mathrm{d}}{\mathrm{d}x}\hat{\alpha}^{(m)}. \tag{2.58}$$

The difference $\hat{L}^{(m)} - L^{(m)}$ does not contain derivatives of u_1 or u_2 and hence does not contribute to the symplectic form. As a consequence, $(\mathrm{d}/\mathrm{d}x)\hat{\alpha}^{(m)} = (\mathrm{d}/\mathrm{d}x)\alpha^{(m)}$, and the symplectic form is the same as for the homogeneous case.

Given $\hat{L}^{(m)}$, the corresponding Hamiltonian $\hat{H}^{(m)}$ can be constructed from

$$\frac{\mathrm{d}}{\mathrm{d}x}\hat{H}^{(m)} = -\sum_{i=1}^{2} u_i' \frac{\delta\hat{L}^{(m)}}{\delta u_i} - \frac{\partial}{\partial x}\hat{L}^{(m)}. \tag{2.59}$$

We obtain

$$\hat{H}^{(m)} = -(m+1)[\mathrm{Tr}(DR_{m+2} + UR_{m+1}) - (x + c/a)u_1u_2]. \tag{2.60}$$

It differs from Dickey's Hamiltonian by the term $(m+1)(x + c/a)u_1u_2$.

3. Isomonodromic deformation interpreted in terms of the Hamiltonian structure

We define

$$A^{(m)} \equiv A^{(m)}(U(x), x; \zeta) = \tilde{\tilde{R}}^{(m)} \tag{3.1}$$

and are going to investigate the IMP connected with

$$(\partial/\partial\zeta) Y = A^{(m)}(\zeta) Y. \tag{3.2}$$

We expect the deformation equation of (3.1) to be

$$(\partial/\partial x) A^{(m)} + D + [U + \zeta D, A^{(m)}] = 0 \tag{3.3}$$

and in § 3.2 this will actually be proved. Equation (3.3) is equivalent to $[D, \hat{R}_{m+1}] = 0$ and therefore it may be interpreted as the Euler-Lagrange equation of a Hamiltonian system. Furthermore, the exponent of local monodromy, T_0, and the 1-form ω defined in the appendix can be identified with quantities of the Hamiltonian system.

3.1. Calculation of $T^{(m)}$

Following the notation of the appendix we state the following proposition.

Proposition 1. For all $m \in \mathbb{N}$:

(i)

$$T^{(m)}(\zeta) = -\frac{1}{m+1} T^{(m)}_{-m-1} \zeta^{m+1} - T^{(m)}_{-1} \zeta + T^{(m)}_0 \log \frac{1}{\zeta},$$

$$T^{(m)}_{-m-1} = -\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \qquad T^{(m)}_{-1} = \begin{pmatrix} ax+c & 0 \\ 0 & -ax-c \end{pmatrix},$$

$$T^{(m)}_0 = (R_{m+1})_{\mathrm{D}}. \tag{3.4}$$

(ii) For all i, $p \leqslant i \leqslant m$,

$$F^{(m)}_i = \Phi_i. \tag{3.5}$$

The proof of proposition 1 is essentially based on the following three lemmas. A direct consequence of the definition of Ψ by $\Phi(\zeta)\Psi(\zeta) = 1$ is lemma 1.

Lemma 1. For all $n \in \mathbb{N}$

$$\sum_{g=0}^{n} (\varphi^{i1}_g \Psi^{1j}_{n-g} + \varphi^{i2}_g \Psi^{2j}_{n-g}) = 0 \tag{3.6}$$

where $(i, j) \in \{(1,1), (1,2), (2,1), (2,2)\}$. It is readily seen that for all $i \in \mathbb{N}$

$$\Phi_i = (\Phi_i)_{\mathrm{OD}}, \qquad \Psi^{11}_i = \Psi^{22}_i. \tag{3.7}$$

Herewith, and by taking advantage of lemma 1, we easily prove lemma 2.

Lemma 2. For all $n \in \mathbb{N}$

$$\left(\sum_{g=1}^{n} R_g \Phi_{n-g} \right)_D = 0. \tag{3.8}$$

Lemma 3. For all $n \in \mathbb{N}$

$$\left(\sum_{g=1}^{n} R_g \Phi_{n-g}\right)_{\substack{12\\21}} = \mp(b_1 - b_2)\varphi^{\substack{12\\21}}. \tag{3.9}$$

The proof of proposition 1 is now carried out by first calculating $T^{(m)}_{-m-1}$, $T^{(m)}_{-m}$ and $F^{(m)}_1$ and then concluding by induction from $T^{(m)}_{-m}, \ldots, T^{(m)}_{-g}$, $F^{(m)}_1, \ldots, F^{(m)}_{m-g+1}$ to $T^{(m)}_{-m}, \ldots, T^{(m)}_{-g+1}$, $F^{(m)}_1, \ldots, F^{(m)}_{m-g+2}$ for $g = m, \ldots, 3$. Finally $T^{(m)}_{-1}$, $F^{(m)}_m$ and $T^{(m)}_0$ are computed.

3.2. *The deformation equations as Hamilton's canonical equations of motion*

In the introduction of § 3 we promised to prove that $(\partial/\partial x)A^{(m)} + D + [U + \zeta D, A^{(m)}] = 0$ is actually the deformation equation of (3.2). According to the appendix we have to show that the 1-form $\check{\Omega}^{(m)} = \bar{\check{\Omega}}^{(m)}\, \mathrm{d}x$ is identical to $\hat{\Omega} = -(U + \zeta D)\, \mathrm{d}x$.

We have found two ways to prove $\check{\Omega} = \hat{\Omega}$. First, we apply the procedure given by (A14), (A15) to construct the deformation equation. From proposition 1, $\mathrm{d}'T^{(m)} = -\mathrm{D}\zeta\, \mathrm{d}x$; furthermore $[D, F^{(m)}_1] = -U$. Thus $\check{\Omega}^{(m)}$ turns out to depend only apparently on m:

$$\check{\Omega}^{(m)}(\zeta) = \check{\Omega}(\zeta) = -(U + \zeta D)\, \mathrm{d}x \equiv \hat{\Omega}(\zeta). \tag{3.10}$$

In § 3.3 we shall show that from

$$(\partial/\partial \zeta)\, Y^{(m)}_j = A^{(m)}\, Y^{(m)}_j \tag{3.11}$$

and

$$(\partial/\partial x) A^{(m)} = (\partial/\partial \zeta)\bar{\hat{\Omega}} + [\bar{\hat{\Omega}}, A^{(m)}] \tag{3.12}$$

we can conclude

$$(\partial/\partial x)\, Y^{(m)}_j = \bar{\hat{\Omega}}\, Y^{(m)}_j. \tag{3.13}$$

In proposition 5, the invariance of the Stokes matrices will be deduced explicitly from (3.13).

The second proof of $\check{\Omega} = \hat{\Omega}$ is likewise based on (3.13). Comparing $(\partial/\partial x)\, Y_j = \bar{\hat{\Omega}}\, Y_j$ with the deformation equation (A14) we get $(\bar{\check{\Omega}} - \bar{\hat{\Omega}})\, Y^{(m)}_j = 0$. Since $Y^{(m)}_j$ is invertible, this is equivalent to

$$\check{\Omega} = \hat{\Omega}. \tag{3.14}$$

3.3. *Conclusions from the deformation equation, invariance of the monodromy data*

For a 2×2 matrix M let

$$|M| = \sup_{\|x\|=1} \|M \cdot x\|, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \|x\| = |x_1 + \mathrm{i}x_2|, \qquad x_1, x_2 \in \mathbb{C}, \tag{3.15}$$

with

$$|M_1 + M_2| \leqslant |M_1| + |M_2|, \tag{3.16}$$

$$|M_1 \cdot M_2| \leqslant |M_1| \cdot |M_2|. \tag{3.17}$$

Proposition 2. Let Y be a fundamental solution of $(\partial/\partial\zeta)\, Y = A(\zeta)\, Y$, $A(\zeta)$ a 2×2-matrix-valued polynomial in ζ. Following the notation of the appendix, let $Y(\zeta) \sim$

$\hat{Y}(\zeta)\,e^{T(\zeta)}$ for $\zeta\to\infty$ in a sector $\mathscr{S}$ with centre at $\zeta=\infty$. If Ω solves $(\partial/\partial x)A=(\partial/\partial\zeta)\Omega+[\Omega, A]$ and for every $\varepsilon>0$ there exists a ζ_0 such that for every $\bar{\zeta}\in\mathscr{S}$ with $|\bar{\zeta}|\geq|\zeta_0|$

$$|(\partial/\partial x)\,Y-\Omega\,Y|_{|\zeta=\bar{\zeta}}\leq\varepsilon\cdot|e^{T(\bar{\zeta})}|, \tag{3.18}$$

then

$$\frac{\partial}{\partial x}Y-\Omega\,Y=0 \tag{3.19}$$

for all ζ.

Proof. By assumption

$$\frac{\partial}{\partial x}Y=Y\left(Y^{-1}\frac{\partial}{\partial x}Y\right)_{|\zeta=\zeta_0}+Y\int_{\zeta_0}^{\zeta}\frac{\partial}{\partial\zeta}\left(Y^{-1}\frac{\partial}{\partial x}Y\right)\mathrm{d}\zeta \tag{3.20}$$

is transformed to

$$(\partial/\partial x)\,Y=\Omega\,Y+Y\{Y^{-1}[(\partial/\partial x)\,Y-\Omega\,Y]\}_{|\zeta=\zeta_0}. \tag{3.21}$$

Decomposing $Y(\zeta)^{-1}=e^{-T(\zeta)}\bar{Y}(\zeta)^{-1}$ and applying (3.16), (3.17) we obtain

$$|(\partial/\partial x)\,Y-\Omega\,Y|\leq|Y|(|\bar{Y}^{-1}|\cdot|e^{-T}|\cdot\varepsilon\cdot|e^{T}|)=|Y|(|\bar{Y}^{-1}|\cdot\varepsilon)_{|\zeta=\bar{\zeta}} \tag{3.22}$$

for every $\bar{\zeta}\in\mathscr{S}$ with $|\bar{\zeta}|\geq|\zeta_0|$.

From the asymptotic expansion $\bar{Y}(\zeta)^{-1}\sim\hat{Y}(\zeta)^{-1}$ for $\zeta\to\infty$ in $\mathscr{S}$ it can be deduced that $|\bar{Y}(\bar{\zeta})^{-1}|$ remains finite as $\bar{\zeta}\to\infty$. Thus $(\partial/\partial z)\,Y(\zeta)=\Omega(\zeta)\,Y(\zeta)$ pointwise in ζ.

The next proposition is necessary for the proof of proposition 4, but it is especially interesting in itself.

Proposition 3. If $(\partial/\partial x)A^{(m)}+D+[U+\zeta D, A^{(m)}]=0$, then

$$(\partial/\partial x)\,T_0^{(m)}=(\partial/\partial x)(R_{m+1})_{\mathrm{D}}=0. \tag{3.23}$$

Proof. $(\partial/\partial x)A^{(m)}+D+[U+\zeta D, A^{(m)}]=0$ is equivalent to

$$[D, \hat{R}_{m+1}]=0, \qquad \text{i.e. } (\hat{R}_{m+1})_{\mathrm{OD}}=0.$$

So from

$$R'_{m+1}=[U, R_{m+1}]=-[D, R_{m+2}]$$

we get

$$\begin{aligned}0&=(R'_{m+1})_{\mathrm{D}}+[U,(R_{m+1})_{\mathrm{OD}}]\\&=(R'_{m+1})_{\mathrm{D}}+[U,(\hat{R}_{m+1})_{\mathrm{OD}}+(x+c/a)\,U]\\&=(R'_{m+1})_{\mathrm{D}}=T_0^{(m)\prime}.\end{aligned} \tag{3.24}$$

Now, the solutions $Y_j^{(m)}$ defined in the appendix fulfil the assumptions of proposition 2.

Proposition 4. If $(\partial/\partial x)A^{(m)}+D+[U+\zeta D, A^{(m)}]=0$, then for every $\varepsilon>0$ there exists a ζ_0 such that for all $m\in\mathbb{N}$, all $j\in\mathbb{Z}$ and for every $\bar{\zeta}\in\mathscr{S}_j^{(m)}$ with $|\bar{\zeta}|\geq|\zeta_0|$,

$$|(\partial/\partial x)\,Y_j^{(m)}-\hat{\Omega}\,Y_j^{(m)}|_{|\zeta=\bar{\zeta}}\leq\varepsilon\cdot|\exp[T^{(m)}(\bar{\zeta})]|. \tag{3.25}$$

Proof. Let j and m be fixed.

$$Y_j^{(m)}(\zeta) \sim \hat{Y}^{(m)}(\zeta)\exp[T^{(m)}(\zeta)]$$

for $\zeta \to \infty$ in $\mathscr{S}_j^{(m)}$, where the series $\hat{Y}^{(m)}$ is in general divergent. To avoid the problem which may be caused by this divergence, consider the quantity

$$Y^{(m),l}(\zeta) := \left(\sum_{k=0}^{l} \hat{Y}_k^{(m)}\zeta^{-k}\right)\exp[T^{(m)}(\zeta)]. \tag{3.26}$$

We write

$$\frac{\partial}{\partial x}Y_j^{(m)} - \bar{\hat{\Omega}}Y_j^{(m)}$$

$$= \left(\frac{\partial}{\partial x}Y_j^{(m)} - \frac{\partial}{\partial x}Y^{(m),l}\right) - \bar{\hat{\Omega}}(Y_j^{(m)} - Y^{(m),l}) + \left(\frac{\partial}{\partial x}Y^{(m),l} - \bar{\hat{\Omega}}Y^{(m),l}\right) \tag{3.27}$$

and try to bound the norm of each of the three terms on the right-hand side by $\frac{1}{3}\varepsilon \cdot |e^{T(m)}|$.

(i) By calculating the first terms of $Y^{(m),l}$ and by $(R'_{m+1})_{\mathrm{D}} = 0$, $|(\partial/\partial x)Y^{(m),l} - \bar{\hat{\Omega}}Y^{(m),l}|_{|\zeta=\bar{\zeta}}$ for all $\bar{\zeta} \in \mathscr{S}_j^{(m)}$ with $|\bar{\zeta}| > |\zeta_1|$ is estimated by $\frac{1}{3}\varepsilon \cdot |\exp[T^{(m)}(\bar{\zeta})]|$ for a suitably chosen ζ_1.

(ii) By making use of the asymptotic expansions of $Y_j^{(m)}$ and $(\partial/\partial x)Y_j^{(m)}$, one can find ζ_2 and ζ_3 with $|\zeta_3| > |\zeta_2| > |\zeta_1|$ such that the remaining two terms can be bounded in the same way. Taking $\zeta_0 := \zeta_3$, the assertion is proved.

As a conclusion of the last three propositions, we have the following proposition.

Proposition 5. If

$$(\partial/\partial x)A^{(m)} = (\partial/\partial\zeta)\bar{\hat{\Omega}} + [\bar{\hat{\Omega}}, A^{(m)}]$$

then

$$(\partial/\partial x)S_j^{(m)} = 0. \tag{3.28}$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x}S_j^{(m)} &= \frac{\partial}{\partial x}Y_j^{(m)-1}Y_{j+1}^{(m)} + Y_j^{(m)-1}\frac{\partial}{\partial x}Y_{j+1}^{(m)} \\
&= -Y_j^{(m)-1}\left(\frac{\partial}{\partial x}Y_j^{(m)}\right)Y_j^{(m)-1}Y_{j+1}^{(m)} + Y_j^{(m)-1}\frac{\partial}{\partial x}Y_{j+1}^{(m)} \\
&= -Y_j^{(m)-1}\Omega Y_{j+1}^{(m)} + Y_j^{(n)-1}\Omega Y_{j+1}^{(m)} \\
&= 0.
\end{aligned} \tag{3.29}$$

3.4. *Equivalence of $\omega^{(m)}$ and $\hat{H}^{(m)}$*

The 1-form

$$\omega = \sum_{\mu=1,\dots,n,\infty} -\operatorname*{Res}_{\zeta=a_\mu} \operatorname{Tr}\left(\hat{Y}^{(\mu)-1}\frac{\partial\hat{Y}^{(\mu)}}{\partial\zeta}\mathrm{d}'T^{(\mu)}\right)$$

(notice the different meaning of indices $^{(\mu)}$ and $^{(m)}$) is a fundamental element of the theory of isomonodromic deformations.

Jimbo *et al* (1979) notice that in the case where the set of deformation data is restricted to $\{a_1, \ldots, a_n, T^{(\infty)}_{-1}\}$, the deformation equations can be derived as canonical equations of motion from a Hamiltonian which is closely related to the 1-form ω of this special case. However, they do not develop the complete symplectic framework.

Furthermore they remark that ω is an important link between IMPs and other—at first sight unrelated—topics of mathematical physics. For example, there is a close connection between the n-point correlation function of the two-dimensional Ising model and the 1-form ω of a certain IMP.

Now, we want to show that for our IMP ($\mu = \infty$), $\bar{\omega}^{(m)}$ is identical to the Hamiltonian $\hat{H}^{(m)}$ up to a constant factor. From (A17)

$$\omega^{(m)} = \bar{\omega}^{(m)}\,\mathrm{d}x = -\operatorname*{Res}_{\zeta=\infty} \operatorname{Tr} \hat{Y}^{(m)}(\zeta)^{-1} \frac{\partial \hat{Y}^{(m)}(\zeta)}{\partial \zeta}\,\mathrm{d}' T^{(m)}(\zeta). \tag{3.30}$$

The coefficient of ζ^{-1} in $\hat{Y}^{(m)}(\partial \hat{Y}^{(m)}/\partial\zeta)\,\mathrm{d}'T^{(m)}$ is easily calculated to be $(F_1^{(m)} + D_1^{(m)})D\,\mathrm{d}x$. Since $\operatorname{Tr}\{(F_1^{(m)} + D_1^{(m)})D\,\mathrm{d}x\} = \operatorname{Tr}\{D_1^{(m)}D\,\mathrm{d}x\}$ we obtain

$$\bar{\omega}^{(m)}(U(x), x) = -\operatorname{Tr} D_1^{(m)} D. \tag{3.31}$$

To identify $\bar{\omega}^{(m)}$ and $\hat{H}^{(m)}$ we have to show the following.

Proposition 6.

$$\operatorname{Tr} D_1^{(m)} D = \operatorname{Tr}\{DR_{m+2} + UR_{m+1}\} - (x + c/a)u_1u_2. \tag{3.32}$$

The proof is somewhat technical and may be omitted. It is essentially based on Lemmas 2 and 3 and the calculation of $D_1^{(m)}$ and $F_{m+1}^{(m)}$ following the procedure of (A7).

Corollary.

$$\bar{\omega}^{(m)} = (m+1)^{-1}\hat{H}^{(m)}. \tag{3.33}$$

3.5. *Example* $m = 2$

In order to illustrate our results obtained so far, let us discuss an example. Let $m = 2$

$$A^{(2)}(\zeta) = -\sum_{j=-3}^{-1} A_{-j}\zeta^{-j-1}. \tag{3.34}$$

Then our general procedure yields

$$\begin{aligned} A_{-3} &= -\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \qquad A_{-2} = -\begin{pmatrix} 0 & (b/a)u_1 \\ (b/a)u_2 & 0 \end{pmatrix}, \\ A_{-1} &= -\begin{pmatrix} -(b/2a^2)u_1u_2 - ax - c & -(b/2a^2)u_1' \\ (b/2a^2)u_2' & (b/2a^2)u_1u_2 + ax + c \end{pmatrix}. \end{aligned} \tag{3.35}$$

Thus we obtain the IMP associated to

$$\frac{\partial}{\partial\zeta} Y = \begin{pmatrix} b\zeta^2 - \dfrac{b}{2a^2}u_1u_2 - ax - c & \dfrac{b}{a}u_1\zeta - \dfrac{b}{2a^2}u_1' \\ \dfrac{b}{a}u_2\zeta + \dfrac{b}{2a^2}u_2' & -b\zeta^2 + \dfrac{b}{2a^2}u_1u_2 + ax + c \end{pmatrix} Y. \tag{3.36}$$

The deformation equations are $(\partial/\partial x)A^{(2)}+D+[U+\zeta D, A^{(2)}]=-[D, \hat{R}_3]=0$, i.e.

$$\hat{R}_3^{12}=(b/4a^3)(u_1''-2u_1^2u_2)-(x+c/a)u_1=0,$$
$$\hat{R}_3^{21}=(b/4a^3)(u_2''-2u_1u_2^2)-(x+c/a)u_2=0. \tag{3.37}$$

In the case of $b/4a^3=1$, $c=0$, $u_1=u_2=q$, we obtain

$$q''=2q^3+qx, \tag{3.38}$$

the Painlevé II equation $q''=2q^3+qx+\nu$ with $\nu=0$. One calculates

$$T^{(2)}(\zeta)=\left[\tfrac{1}{3}b\zeta^3-(ax+c)\zeta+\frac{b}{4a^3}(u_1'u_2-u_1u_2')\log\left(\frac{1}{\zeta}\right)\right]\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$
$$D_1^{(2)}(\zeta)=\left[\frac{b}{8a^4}(u_1'u_2'+u_1^2u_2^2)-\frac{1}{2a}\left(x+\frac{c}{a}\right)u_1u_2\right]\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{3.39}$$

Using equation (3.37) one easily checks that $T_0=(b/4a^3)(u_1'u_2-u_1u_2')\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ is independent of x. Further from (3.36), (3.37) and propositions 2–4 we conclude

$$\frac{\partial}{\partial x}Y=-\begin{pmatrix}a\zeta&u_1\\u_2&-a\zeta\end{pmatrix}Y. \tag{3.40}$$

Employing (3.37) once more, the Lagrangian and the Hamiltonian are calculated as (compare equations (2.57) and (2.60))

$$\hat{L}^{(2)}=(b/4a^3)(u_1'u_2'-u_1^2u_2^2)-5(x+c/a)u_1u_2, \tag{3.41}$$

$$\hat{H}^{(2)}=-3[(b/4a^3)(u_1'u_2'-u_1^2u_2^2)-(x+c/a)u_1u_2]. \tag{3.42}$$

Eventually

$$\bar{\omega}^{(2)}=-[(b/4a^3)(u_1'u_2'-u_1^2u_2^2)-(x+c/a)u_1u_2]. \tag{3.43}$$

After having been transformed to an appropriate set of symplectic coordinates, the deformation equations (3.37) can be explicitly formulated as Hamilton's canonical equations derived from the Hamiltonian $\hat{H}^{(2)}$.

The symplectic 2-form $\mathrm{d}\hat{\alpha}^{(2)}$ is given by

$$\mathrm{d}\hat{\alpha}^{(2)}=-3(b/4a^3)(\mathrm{d}u_2'\wedge\mathrm{d}u_1+\mathrm{d}u_1'\wedge\mathrm{d}u_2). \tag{3.44}$$

Comparing with the canonical expression

$$\mathrm{d}\hat{\alpha}^{(2)}=\mathrm{d}p_1\wedge\mathrm{d}q_1+\mathrm{d}p_2\wedge\mathrm{d}q_2 \tag{3.45}$$

of the symplectic form, a possible choice of the symplectic coordinates turns out to be

$$p_1=-\frac{3b}{4a^3}u_2',\qquad q_1=u_1,\qquad p_2=-\frac{3b}{4a^3}u_1',\qquad q_2=u_2. \tag{3.46}$$

The transformed Hamiltonian is

$$\hat{H}^{(2)}=\hat{H}^2(p_1,p_2;q_1,q_2)=-\frac{4a^3}{3b}p_1p_2+\frac{3b}{4a^3}q_1^2q_2^2+3\left(x+\frac{c}{a}\right)q_1q_2 \tag{3.47}$$

and the canonical equations of motion $q_i' = \partial H/\partial p_i$, $p_i' = -\partial H/\partial q_i$, $i = 1, 2$, become

$$q_1' = -(4a^3/3b)p_2, \qquad q_2' = -(4a^3/3b)p_1, \tag{3.48}$$

$$\begin{aligned} p_1' &= -(3b/2a^3)q_1q_2^2 - 3(x + c/a)q_2, \\ p_2' &= -(3b/2a^3)q_1^2q_2 - 3(x + c/a)q_1. \end{aligned} \tag{3.49}$$

The equations (3.48) are solved by (3.46), yielding the identities $u_1' = u_1'$, $u_2' = u_2'$, respectively, whereas after substitution with (3.46) the equations (3.49) become equivalent to the deformation equations (3.37).

4. Stokes matrices

4.1. *Stokes matrices as first integrals of the Hamiltonian structure*

Apart from $(R_{m+1})_D$ the first integrals of Dickey's Hamiltonian systems are no longer invariant with respect to the new equations of motion (= deformation equations)

$$(\partial/\partial x)A^{(m)} + D + [U + \zeta D, A^{(m)}] = -[D, \hat{R}_{m+1}] = 0. \tag{4.1}$$

On the other hand, these equations yield the elements of the Stokes matrices as new first integrals.

It could be that the Stokes matrices are functions of R_{m+1}^{11} and R_{m+1}^{22} alone, or even identically constant. In these cases, of course, the Stokes matrices would not yield any interesting new first integrals of $[D, \hat{R}_{m+1}] = 0$. To exclude this possibility, we should like to know the explicit form of the Stokes matrices, in order to show that they are functionally independent of $(R_{m+1})_D$.

4.2. *Problems*

The general problem, to evaluate the set of Stokes matrices for a given differential equation $(\partial/\partial\zeta)Y = A(\zeta)Y$, $A(\zeta)$ rational in ζ, is still unsolved. Even for special cases of the coefficient matrix $A(\zeta)$, we could hardly find any results in the literature concerning the corresponding Stokes matrices.

The inverse problem, however, of constructing a differential equation with given singularities and Stokes matrices has been discussed by Sibuya (1975b, 1977). Sibuya (1975a) also investigates the structure of the Stokes multipliers of a second-order differential equation corresponding to the Stokes matrices of first-order systems.

4.3. *Calculation of the Stokes matrices in the case $m = 1$*

Employing Sibuya's ideas, we succeeded in evaluating the Stokes matrices for $m = 1$. They prove to be non-trivial and functionally independent of $-\hat{R}_2^{11} = \hat{R}_2^{22} = u_1u_2$. It can be strongly conjectured that this result is not restricted to $m = 1$, but is a general feature of the Stokes matrices.

For $m = 1$, (3.2) becomes

$$\frac{d}{d\zeta}Y = \begin{pmatrix} b\zeta - ax - c & (b/a)u_1 \\ (b/a)u_2 & -b\zeta + ax + c \end{pmatrix} Y. \tag{4.2}$$

The deformations (3.3) are equivalent to

$$0=-\frac{b}{2a^2}u_1'-\left(x+\frac{c}{a}\right)u_1, \qquad 0=\frac{b}{2a^2}u_2'-\left(x+\frac{c}{a}\right)u_2. \tag{4.3}$$

To adapt our equations to the formal structure of Sibuya's equation, we set $b=1$.

The solutions $Y_j^{(1)}$ of (4.2) and the corresponding sectors $\mathcal{S}_j^{(1)}$, $j=0,1,2,3$, are defined in the appendix. The asymptotic expansion of $Y_j^{(1)}(\zeta)$ for $\zeta\to\infty$ in $\mathcal{S}_1^{(1)}$ is given by the formal solution $Y^{(\infty)(1)}(\zeta)=\hat{Y}^{(1)}(\zeta)\exp(T^{(1)}(\zeta))$ of (4.2):

$$\begin{aligned}\begin{pmatrix}Y_j^{(1)11}\\ Y_j^{(1)21}\end{pmatrix}&\sim\begin{pmatrix}1+(1/2a^2)(ax+c)u_1u_2 & \zeta^{-1}+\ldots\\ (1/2a)u_2 & \zeta^{-1}+\ldots\end{pmatrix}\exp[\tfrac{1}{2}\zeta^2-(ax+c)]\zeta^{(1/2a^2)u_1u_2},\\ \begin{pmatrix}Y_j^{(1)12}\\ Y_j^{(1)22}\end{pmatrix}&\sim\begin{pmatrix}-(1/2a)u_1 & \zeta^{-1}+\ldots\\ 1+(1/2a^2)(ax+c)u_1u_2 & \zeta^{-1}+\ldots\end{pmatrix}\exp[-\tfrac{1}{2}\zeta^2+(ax+c)]\zeta^{-(1/2a^2)u_1u_2}.\end{aligned} \tag{4.4}$$

The first-order system (4.2) can be transformed to a single second-order differential equation of the type

$$(\mathrm{d}^2/\mathrm{d}\zeta^2)y-[\zeta^2+a_1(x)\zeta+a_2(z)]y=0 \tag{4.5}$$

discussed by Sibuya. Evaluation of the coefficients yields

$$a_1(x)=-2(ax+c), \qquad a_2(x)=(ax+c)^2+a^{-2}u_1u_2+1. \tag{4.6}$$

Conversely, from every solution of (4.5), (4.6) one can obtain a solution of

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\begin{pmatrix}y\\ z\end{pmatrix}=A^{(1)}(\zeta)\begin{pmatrix}y\\ z\end{pmatrix} \tag{4.7}$$

by putting $z:=(a/u_1)[y'-(\zeta-ax-c)y]$. Thus, (4.5), (4.6) and (4.7) are equivalent.

The results of Sibuya's monograph (1975a) which are relevant to us can be summarised in the following statement. Sibuya treats the equation

$$(\mathrm{d}^2/\mathrm{d}\zeta^2)y-(\zeta^\mu+a_1\zeta^{\mu-1}+\ldots+a_{\mu-1}\zeta+a_\mu)y=0 \tag{4.8}$$

and shows the following.

(i) The differential equation (4.8) has a unique solution

$y=y_\mu(\zeta;a_1,\ldots,a_\mu)\equiv y_\mu(\zeta,\alpha)$, such that:

(a) $y_\mu(\zeta,\alpha)$ is an entire function of $(\zeta;a_1,\ldots,a_\mu)$;

(b) $y_\mu(\zeta,\alpha)$ admits an asymptotic representation (in the sense of Wasov (1965)):

$$y_\mu(\zeta,\alpha)\sim\zeta^{r_\mu}\left(1+\sum_{n=1}^{\infty}B_{\mu n}\zeta^{-n/2}\right)\exp[-E_\mu(\zeta,\alpha)] \tag{4.9}$$

as ζ tends to infinity in any closed subsector of the open sector $|\arg\zeta|<3\pi/(\mu+2)$, where

$$E_\mu(\zeta,\alpha)=\frac{2}{\mu+2}\zeta^{(\mu+2)/2}+\sum_{n=1}^{\mu+1}A_{\mu,n}\zeta^{(\mu+2-m)/2} \tag{4.10}$$

and r_μ, $A_{\mu,n}$, $B_{\mu,n}$ are polynomials in $(a_1,\ldots,a_\mu)$.

If we put

$$(1+a_1\zeta^{-1}+\ldots+a_\mu\zeta^{-\mu})^{1/2}:=1+\sum_{k=1}^{\infty}b_k\zeta^{-k} \tag{4.11}$$

then the quantities r_μ and $A_{\mu,n}$ are given, respectively, by

$$r_\mu = \begin{cases} -\frac{1}{4}\mu, & \mu \text{ odd}, \\ -\frac{1}{4}\mu - b_{\mu/2+1}, & \mu \text{ even}, \end{cases} \tag{4.12}$$

and

$$\sum_{n=1}^{\mu+1} A_{\mu,n}\zeta^{(\mu+2-n)/2} = \sum_{1\leq h\leq \mu/2+1} \frac{2}{\mu+2-2h} b_h \zeta^{(\mu+2-2h)/2}. \tag{4.13}$$

(ii) A change in the independent variable ζ by $\hat{\zeta} = e^{i\varphi}\zeta$, $\varphi\in\mathbb{R}$ yields

$$\begin{gathered} \frac{d^2}{d\hat{\zeta}^2}y - \exp[-i(\mu+2)\varphi]Q(\hat{\zeta})y = 0, \\ Q(\hat{\zeta}) = \hat{\zeta}^\mu + \sum_{h=1}^{\mu} e^{ih\varphi} a_h \hat{\zeta}^{\mu-h}. \end{gathered} \tag{4.14}$$

Therefore, if we choose φ such that $\exp[i(\mu+2)\varphi] = 1$, the function $y = y_\mu(\hat{\zeta}; e^{i\varphi}a_1, \ldots, e^{i\mu\varphi}a_\mu)$ is a solution of (4.14). This means, if we set

$$\theta = \exp[i2\pi/(\mu+2)] \tag{4.15}$$

and

$$y_{\mu,j}(\zeta_i a_1, \ldots, a_\mu) = y_\mu(\theta^{-j}\zeta; \theta^{-j}a_1, \ldots, \theta^{-\mu j}a_\mu), \tag{4.16}$$

$j\in\mathbb{Z}$, the $y_{\mu,j}(\zeta, \alpha)$ are solutions of (4.8). In particular

$$y_{\mu,0}(\zeta, \alpha) = y_\mu(\zeta, \alpha). \tag{4.17}$$

(iii) The sector $\mathcal{S}_j^{(\mu)}$ is defined by $|\arg\zeta - 2j\pi/(\mu+2)| < \pi/(\mu+2)$. If a solution of (4.8) tends to zero (infinity) as ζ tends to infinity along any direction in the sector $\mathcal{S}_j^{(\mu)}$, then this solution is said to be subdominant (dominant) in the sector $\mathcal{S}_j^{(\mu)}$. The solution $y_{\mu,j}$ is subdominant in $\mathcal{S}_j^{(\mu)}$ and dominant in $\mathcal{S}_{j-1}^{(\mu)}$ and $\mathcal{S}_{j+1}^{(\mu)}$.

(iv) The function $y_{\mu,j}$ admits an asymptotic representation

$$y_{\mu,j}(\zeta, \alpha) \sim \theta^{-jr_{\mu,j}}\zeta^{r_{\mu,j}}\left(1 + \sum_{n=1}^{\infty} B_{\mu,n;j}\zeta^{-m/2}\right)\exp[(-1)^{j+1}E_\mu(\zeta, \alpha)] =: y_{\mu,j}^{(\infty)} \tag{4.18}$$

as ζ tends to infinity in any closed subsector of the open sector $\mathcal{S}_{j-1}^{(\mu)} \cup \overline{\mathcal{S}_j^{(\mu)}} \cup \mathcal{S}_{j+1}^{(\mu)}$ where

$$r_{\mu,j} = \begin{cases} -\frac{1}{4}\mu, & \mu \text{ odd}, \\ -\frac{1}{4}\mu + (-1)^{j+1}b_{\mu/2+1}, & \mu \text{ even}. \end{cases} \tag{4.19}$$

(v) The two solutions $y_{\mu,j+1}$ and $y_{\mu,j+2}$ are linearly independent because $y_{\mu,j+1}$ is subdominant in $\mathcal{S}_{j+1}^{(\mu)}$ whereas $y_{\mu,j+2}$ is dominant here. Therefore, $y_{\mu,j}$ is a linear combination of $y_{\mu,j+1}$ and $y_{\mu,j+2}$. Set

$$y_{\mu,j}(\zeta, \alpha) = c_j(\alpha)y_{\mu,j+1}(\zeta, \alpha) + \tilde{c}_j(\alpha)y_{\mu,j+2}(\zeta, \alpha). \tag{4.20}$$

Remark. The quantities c_j and $\tilde{c}_j$ are called Stokes multipliers. As we shall see, the essential part of the Stokes matrices we are looking for is given by $c_j(\alpha)$.

(vi) In the case $\mu = 2$:

$$c_j(a_1, a_2) = \begin{cases} 2^{b_2}\exp[\frac{1}{4}a_1^2 - i\pi(\frac{1}{2}b_2 - \frac{1}{4})]\dfrac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}+b_2)}, & j \text{ even}, \\ 2^{-b_2}\exp[-\frac{1}{4}a_1^2 + i\pi(\frac{1}{2}b_2 + \frac{1}{4})]\dfrac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}-b_2)}, & j \text{ odd}, \end{cases} \tag{4.21}$$

$$\tilde{c}_j(a_1, a_2) = \begin{cases} (-\mathrm{i}) \exp(-\mathrm{i}\pi b_2), & j \text{ even}, \\ (-\mathrm{i}) \exp(\mathrm{i}\pi b_2), & j \text{ odd}, \end{cases} \tag{4.22}$$

where $b_2 = \frac{1}{2}a_2 - \frac{1}{8}a_1^2$.

Applying the above results to our equation (4.5), (4.6), we obtain:

$$\begin{aligned} &\theta = \mathrm{i}, \qquad b_1(x) = -(ax+c), \qquad b_2(x) = \tfrac{1}{2}(1 + a^{-2}u_1u_2), \\ &r_{2,j} = \begin{cases} -1 - (2a^2)^{-1}u_1u_2, & j \text{ even}, \\ (2a^2)^{-1}u_1u_2, & j \text{ odd}, \end{cases} \\ &A_{2,1} = 0, \qquad A_{2,2} = -(ax+c), \qquad A_{2,3} = 0, \\ &E_2 = \tfrac{1}{2}\zeta^2 - (ax+c)\zeta. \end{aligned} \tag{4.23}$$

Following a procedure given by Sibuya, $B_{\mu,n;j}$ can be computed recursively. For $\mu = 2$ we first of all get $B_{2,1;j} = 0$ for all $j \in \mathbb{Z}$; therefore $B_{2,k;j}$ turns out to be zero for all $k \in \mathbb{N}$, k odd and all $j \in \mathbb{Z}$. Furthermore,

$$B_{2,2;j} = \begin{cases} (1 + (2a^2)^{-1}u_1u_2)(ax+c), & j \text{ even}, \\ (2a^2)^{-1}u_1u_2(ax+c), & j \text{ odd}. \end{cases} \tag{4.24}$$

So $\{1 + \Sigma_{n=1}^{\infty} B_{2,n;j}\zeta^{-n/2}\}$ is actually of the form $\{1 + \Sigma_{n=1}^{\infty} B_{2,2n;j}\zeta^{-n}\}$.

We now argue as follows. In a formal solution $(1 + \Sigma_{j=1}^{\infty} a_j\zeta^{-j})\zeta^r \mathrm{e}^{t(\varphi)}$ of (4.5) the coefficients a_j are uniquely determined. The formal series $(Y^{(\infty)(1)})_{11}$ and $(Y^{(\infty)(1)})_{12}$ and $y_{2,j}^{(\infty)}$ are such formal solutions of (4.5), (4.6). Applying (iv) we obtain:

$$\begin{aligned} y_{2,-1}^{(\infty)} &= \mathrm{i}^{(1/2a^2)u_1u_2}\zeta^{(1/2a^2)u_1u_2}[1 + (1/2a^2)u_1u_2(ax+c)\zeta^{-1} + \mathrm{O}(\zeta^{-2})] \\ &\quad \times \exp[\tfrac{1}{2}\zeta^2 - (ax+c)\zeta], \\ y_{2,0}^{(\infty)} &= \zeta^{-1-(1/2a^2)u_1u_2}\{1 + [1 + (1/2a^2)u_1u_2](ax+c)\zeta^{-1} + \mathrm{O}(\zeta^{-2})\} \\ &\quad \times \exp[-\tfrac{1}{2}\zeta^2 + (ax+c)\zeta], \\ y_{2,1}^{(\infty)} &= \mathrm{i}^{-(1/2a^2)u_1u_2}\zeta^{(1/2a^2)u_1u_2}[1 + (1/2a^2)u_1u_2(ax+c)\zeta^{-1} + \mathrm{O}(\zeta^{-2})] \\ &\quad \times \exp[\tfrac{1}{2}\zeta^2 - (ax+c)\zeta], \\ y_{2,2}^{(\infty)} &= \mathrm{i}^{2+(1/a^2)u_1u_2}\zeta^{-1-(1/2a^2)\mu_1u_2}\{1 + [1 - (1/2a^2)u_1u_2](ax+c)\zeta^{-1} + \mathrm{O}(\zeta^{-2})\} \\ &\quad \times \exp[-\tfrac{1}{2}\zeta^2 + (ax+c)\zeta]. \end{aligned} \tag{4.25}$$

Comparing with

$$\begin{aligned} (Y^{(\infty)(1)})_{11} &= \zeta^{(1/2a^2)u_1u_2}[1 + (1/2a^2)u_1u_2(ax+c)\zeta^{-1} + \mathrm{O}(\zeta^{-2})] \\ &\quad \times \exp[\tfrac{1}{2}\zeta^2 - (ax+c)\zeta], \\ (Y^{(\infty)(1)})_{12} &= -(1/2a)u_1\zeta^{-1}\zeta^{-(1/2a^2)u_1u_2}\{1 + [1 + (1/2a^2)u_1u_2](ax+c)\zeta^{-1} + \mathrm{O}(\zeta^{-2})\} \\ &\quad \times \exp[-\tfrac{1}{2}\zeta^2 + (ax+c)\zeta], \end{aligned} \tag{4.26}$$

we conclude

$$\begin{aligned} (Y^{(\infty)(1)})_{11} &= \mathrm{i}^{-(1/2a^2)u_1u_2}y_{2,-1}^{(\infty)} = \mathrm{i}^{(1/2a^2)u_1u_2}y_{2,1}^{(\infty)}, \\ (Y^{(\infty)(1)})_{12} &= -(1/2a)u_1y_{2,0}^{(\infty)}. \end{aligned} \tag{4.27}$$

According to (i) the solutions $y_{2,j}$ having the formal solutions $y_{2,j}^{(\infty)}$ as asymptotic expansions are uniquely determined. Therefore the relations (4.27) connecting the formal solutions $(Y^{(\infty)(1)})_{11}$, $(Y^{(\infty)(1)})_{12}$ and $y_{2,j}^{(\infty)}$ can be taken over to the actual solutions $Y_{l,11}^{(1)}$, $Y_{l,12}^{(1)}$ and $y_{2,j}$.

Now, to specify the index l and to determine the precise relations of $y_{2,j}$ and $Y_{l,11}^{(1)}$, $Y_{l,12}^{(1)}$ we notice that the solutions y_{2j} are defined in the composed sectors $\check{\mathscr{S}}_{j-1}^{(2)} \cup \check{\mathscr{S}}_{j}^{(2)} \cup \check{\mathscr{S}}_{j+1}^{(2)}$, respectively, whereas $Y_{l,11}^{(1)}$ and $Y_{l,12}^{(1)}$ are restricted to a single sector $\check{\mathscr{S}}_{j}^{(2)}$. $Y_{0,11}^{(1)}$ is dominant in $\check{\mathscr{S}}_{0}^{(2)}$, while $Y_{0,12}^{(1)}$ in $\check{\mathscr{S}}_{0}^{(2)}$ and $Y_{1,11}^{(1)}$ in $\check{\mathscr{S}}_{1}^{(2)}$ are subdominant. Comparing with the asymptotic behaviour of $y_{2,1}$, $y_{2,0}$ and $y_{2,-1}$, we conclude that the subdominant solutions $Y_{0,12}^{(1)}$ and $Y_{1,11}^{(1)}$ in $\check{\mathscr{S}}_{0}^{(2)}$ and $\check{\mathscr{S}}_{1}^{(2)}$, respectively, have to be related to $y_{2,0}$ and $y_{2,1}$. The dominant solution $Y_{0,11}^{(1)}$, however, can be linked to $y_{2,1}$ as well as to $y_{2,-1}$ in the sector $\check{\mathscr{S}}_{0}^{(2)}$.

Since we intend to introduce the Stokes multipliers, we try to couple the solutions of three successive sectors and therefore relate $Y_{0,11}^{(1)}$ to $y_{2,-1}$. Thus we have

$$Y_{0,11}^{(1)} = \mathrm{i}^{-(1/2a^2)u_1u_2} y_{2,-1}, \qquad Y_{1,11}^{(1)} = \mathrm{i}^{(1/2a^2)u_1u_2} y_{2,1}, \qquad Y_{0,12}^{(1)} = -(1/2a)u_1 y_{2,0}. \tag{4.28}$$

Following Birkhoff (1909), the Stokes matrices S_j can be shown to be triangular, having the form $\left(\begin{smallmatrix}1 & s_j\\ 0 & 1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1 & 0\\ s_j & 1\end{smallmatrix}\right)$. This can be concluded from the fact that a solution which is subdominant in $\mathscr{S}_j$ can be continued analytically to $\mathscr{S}_{j+1}$ and has the same asymptotic representation there, which is now dominant, whereas on the other hand, a dominant solution in a sector $\mathscr{S}_j$ has to pick up a subdominant component before it can represent a subdominant solution in the neighbouring sector $\mathscr{S}_{j+1}$.

Making use of this result and combining it with the preceding discussion, we can evaluate the Stokes matrix $S_0^{(1)}$ from $Y_1^{(1)} = Y_0^{(1)} S_0^{(1)}$. Since $Y_{0,11}^{(1)}$ is dominant in $\mathscr{S}_0^{(1)} (\simeq \check{\mathscr{S}}_0^{(2)})$ we have

$$Y_{1,11}^{(1)} = Y_{0,11}^{(1)} + S_0(U(x), x)\, Y_{0,12}^{(1)} \tag{4.29}$$

and $S_0^{(1)}$ has the form

$$S_0^{(1)} = \begin{pmatrix} 1 & 0 \\ s_0 & 1 \end{pmatrix}. \tag{4.30}$$

From (v) we know that

$$y_{2,-1} = c_{-1} y_{2,0} + \tilde{c}_{-1} y_{2,1}. \tag{4.31}$$

Inserting $\tilde{c}_{-1}$ from (vi) and substituting with (4.28) we obtain

$$Y_{1,11}^{(1)} = Y_{0,11}^{(1)} + (2a/u_1)c_{-1} \exp[-\mathrm{i}\pi(1/4a^2)u_1u_2]\, Y_{0,12}^{(1)}. \tag{4.32}$$

Comparing (4.31) and (4.32) and substituting c_{-1} according to (vi), we end with

$$s_0(U(x), x) = \frac{2a}{u_1}\mathrm{i} \exp\left[\left(-\frac{1}{2} - \frac{1}{2a^2}u_1u_2\right)\ln 2 - (ax+c)^2\right] \frac{\sqrt{2\pi}}{\Gamma[-(1/2a^2)u_1u_2]}. \tag{4.33}$$

The same result is obtained by applying the above procedure to $Y_{l,21}^{(1)}$, $Y_{l,22}^{(1)}$ and a corresponding equation (4.5).

The Stokes matrices $S_1^{(1)}$, $S_2^{(1)}$ and $S_3^{(1)}$ can be evaluated in a similar way. The results can be summarised as follows:

$$S_0^{(1)} = \begin{pmatrix} 1 & 0 \\ s_0 & 1 \end{pmatrix}, \qquad S_1^{(1)} = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, \qquad S_2^{(1)} = \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}, \qquad S_3^{(1)} = \begin{pmatrix} 1 & s_3 \\ 0 & 1 \end{pmatrix}, \tag{4.34}$$

with

$$s_0 = \mathrm{i}\frac{2a}{u_1}\exp\left[\left(-\frac{1}{2}-\frac{1}{2a^2}u_1u_2\right)\ln 2-(ax+c)^2\right]\frac{\sqrt{2\pi}}{\Gamma[-(1/2a^2)u_1u_2]},$$

$$s_1 = -\frac{a}{u_2}\exp\left[\left(\frac{1}{2}+\frac{1}{2a^2}u_1u_2\right)\ln 2+(ax+c)^2\right]\exp\left(\mathrm{i}\frac{\pi}{2a^2}u_1u_2\right)\frac{\sqrt{2\pi}}{\Gamma[(1/2a^2)u_1u_2]},$$

$$s_2 = -\mathrm{i}\frac{2a}{u_1}\exp\left[\left(-\frac{1}{2}-\frac{1}{2a^2}u_1u_2\right)\ln 2+(ax+c)^2\right]\exp\left(-\mathrm{i}\frac{\pi}{a^2}u_1u_2\right)\frac{\sqrt{2\pi}}{\Gamma[-(1/2a^2)u_1u_2]},$$

$$s_3 = \frac{a}{u_2}\exp\left[\left(\frac{1}{2}+\frac{1}{2a^2}u_1u_2\right)\ln 2+(ax+c)^2\right]\exp\left(\mathrm{i}\frac{3\pi}{2a^2}u_1u_2\right)\frac{\sqrt{2\pi}}{\Gamma[(1/2a^2)u_1u_2]}. \tag{4.35}$$

Remembering (4.3), $b=1$, it can easily be proved that

$$(\partial/\partial x)s_j = 0, \qquad j=0,1,2,3.$$

The result (4.35) can be checked by inserting it into the consistency condition of monodromy (A13)

$$1 = \exp(2\pi\mathrm{i}T_0^{(1)})S_3^{(1)-1}S_2^{(1)-1}S_1^{(1)-1}S_0^{(1)-1}. \tag{4.36}$$

Appendix

The IMP connected with the differential equation

$$(\mathrm{d}/\mathrm{d}\zeta)Y = A(\zeta)Y, \tag{A1}$$

$$A(\zeta) = \sum_{\nu=1}^{n}\sum_{k=0}^{r_\nu}\frac{A_{\nu,-k}}{(\zeta-a_\nu)^{k+1}} - \sum_{k=1}^{r_\infty}A_{\infty,-k}\zeta^{k-1},$$

is described in detail by Jimbo *et al* (1979, 1981) and Jimbo and Miwa (1981a, b).

We confine ourselves to a single irregular singularity at $\zeta=\infty$, setting $r_\infty = m+1$, i.e.

$$A(\zeta) = A^{(m)}(\zeta) = -\sum_{j=-m-1}^{-1}A_j\zeta^{-j-1},\qquad A_{-m-1} = -\begin{pmatrix} b & 0\\ 0 & -b\end{pmatrix}. \tag{A2}$$

Here we summarise the most important definitions and results of Jimbo *et al* (1981) which are relevant to us.

(i) There exists a unique formal power series

$$\hat{Y}^{(m)}(\zeta) = 1+\sum_{j=1}^{\infty}\hat{Y}_j^{(m)}\zeta^{-j} \tag{A3}$$

and diagonal matrices $T_{-m-1}^{(m)},\ldots,T_0^{(m)}$, such that

$$Y^{(\infty)(m)}(\zeta) = \hat{Y}^{(m)}(\zeta)\exp[T^{(m)}(\zeta)], \tag{A4}$$

$$T^{(m)}(\zeta) = \sum_{j=1}^{m+1}T_{-j}^{(m)}\frac{\zeta^j}{(-j)} + T_0^{(m)}\log\left(\frac{1}{\zeta}\right), \tag{A5}$$

satisfies $(\mathrm{d}/\mathrm{d}\zeta)\,Y^{(\infty)(m)} = A^{(m)}Y^{(\infty)(m)}$ identically in ζ.

(ii) $\hat{Y}^{(m)}$ is uniquely factorised into $\hat{Y}^{(m)}(\zeta) = F^{(m)}(\zeta) D^{(m)}(\zeta)$ where

$$\begin{aligned} F^{(m)}(\zeta) &= 1 + \sum_{j=1}^{\infty} F_j^{(m)} \zeta^{-j}, \qquad F_j^{(m)} = (F_j^{(m)})_{\mathrm{OD}}, \\ D^{(m)}(\zeta) &= 1 + \sum_{j=1}^{\infty} D_j^{(m)} \zeta^{-j}, \qquad D_j^{(m)} = (D_j^{(m)})_{\mathrm{D}}. \end{aligned} \tag{A6}$$

The coefficients $F_j^{(m)}$, $D_j^{(m)}$ and $T_j^{(m)}$ can be computed from $A^{(m)}(\zeta)$ by

$$\begin{aligned} &[F_j^{(m)}, A_{-m-1}^{(m)}] \\ &\quad = \sum_{k=1}^{j} (A_{-m-1+k}^{(m)} F_{j-k}^{(m)} - F_{j-k}^{(m)} T_{-m-1+k}^{(m)}), \qquad 0 \leqslant j \leqslant m+1, \\ &[F_{m+1+j}^{(m)}, A_{-m-1}^{(m)}] \\ &\quad = \sum_{k=1}^{j+m+1} (A_{-m-1+k}^{(m)} F_{m+1+j-k}^{(m)} - F_{m+1+j-k}^{(m)} T_{-m-1+k}^{(m)}) \\ &\quad\quad - \sum_{k=0}^{j-1} F_{j-1-k}^{(m)} I_{k+1}^{(m)} - jF_j^{(m)}, \qquad j \geqslant 1, \\ &\quad I_j^{(m)} = jD_j^{(m)} - \sum_{k=1}^{j-1} I_{j-k}^{(m)} D_k^{(m)}, \\ &\quad \frac{\mathrm{d}}{\mathrm{d}\zeta} D^{(m)} D^{(m)-1} = -\sum_{j=1}^{\infty} I_j^{(m)} \zeta^{-j-1}. \end{aligned} \tag{A7}$$

(iii) Let

$$\begin{aligned} \mathscr{S}_j^{(m)} = \bigg\{ \zeta \,\big|\, |\zeta| > \rho, j\frac{\pi}{m+1} - \frac{\pi}{2(m+1)} \\ - \delta \leqslant \arg\zeta \leqslant (j+1)\frac{\pi}{m+1} - \frac{\pi}{2(m+1)}, \rho, \delta > 0 \bigg\}, \qquad j \in \mathbb{Z}. \end{aligned} \tag{A8}$$

For a sufficiently small $\delta > 0$, there exists a unique holomorphic and invertible solution $Y_j^{(m)}$ to (A1), (A2) in $\mathscr{S}_j^{(m)}$ having the asymptotic expansion

$$Y_j^{(m)} \sim Y^{(\infty)(m)} \qquad \text{in } \mathscr{S}_j^{(m)} \tag{A9}$$

in the sense of Wasov (1965).

(iv) There exists a matrix $S_j^{(m)}$, independent of ζ, such that

$$Y_{j+1}^{(m)} = Y_j^{(m)} S_j^{(m)}. \tag{A10}$$

$S_j^{(m)}, j \in \mathbb{Z}$ are called Stokes matrices. Let γ be any closed path encircling $\zeta = \infty$ and Y_γ be the analytic continuation of Y along γ. Since $\zeta = \infty$ is the only singularity of $A^{(m)}$, $Y^{(m)}$ is single valued, i.e.

$$Y_\gamma(\mathrm{e}^{2\pi\mathrm{i}}\zeta) = Y(\zeta). \tag{A11}$$

Therefore, in our special case the monodromy matrix M_γ defined by

$$Y_\gamma = YM_\gamma \tag{A12}$$

is equal to 1 and the consistency condition of monodromy reads

$$1 = M_\gamma = \exp(2\pi \mathrm{i} T_0^{(m)}) S_{2(m+1)-1}^{(m)-1} \dots S_0^{(m)-1}. \tag{A13}$$

(v) The Stokes matrices $S_j^{(m)}$ and the exponent of formal monodromy $T_0^{(m)}$ constitute the set of monodromy data. The deformation parameters t are chosen to be the set $\{T_{-m-1}^{(m)}, \dots, T_{-1}^{(m)}\}$. d denotes the exterior differentiation with respect to some deformation parameters.

Theorem. The monodromy data stay constant, iff there exists a matrix of 1-forms $\check{\Omega}^{(m)}(\zeta)$ depending rationally on ζ such that

$$\mathrm{d} Y_j^{(m)} = \check{\Omega}^{(m)} Y_j^{(m)} \tag{A14}$$

where

$$\check{\Omega}^{(m)} = \sum_{k=0}^{m+1} \Phi_{-k}^{(m)}(t) \zeta^{-k},$$

$$\hat{Y}^{(m)}(\zeta, t)\, \mathrm{d}' T^{(m)}(\zeta, t)\, \hat{Y}^{(m)}(\zeta, t)^{-1} = \sum_{k=-m-1}^{\infty} \Phi_k^{(m)}(t) \zeta^{-k}, \tag{A15}$$

$$\mathrm{d}' T^{(m)}(\zeta) = \sum_{k=1}^{m+1} \mathrm{d} T_{-k}^{(m)} \frac{\zeta^k}{(-k)}.$$

(A14) is equivalent to the nonlinear deformation equation

$$\mathrm{d} A^{(m)} = \partial \check{\Omega}^{(m)} / \partial \zeta + [\check{\Omega}^{(m)}, A^{(m)}]. \tag{A16}$$

In this article $\mathrm{d} = (\partial/\partial x)\, \mathrm{d}x$ and each 1-form ω can be represented as $\omega = \bar{\omega}\, \mathrm{d}x$.

(vi) *Theorem.* For every solution of the deformation equations (A14), (A16) the one-form

$$\omega^{(m)} = -\operatorname*{Res}_{\zeta=\infty} \operatorname{Tr} \hat{Y}^{(m)-1} \frac{\partial \hat{Y}^{(m)}}{\partial \zeta} \mathrm{d}' T^{(m)} \tag{A17}$$

is closed, $\mathrm{d}\omega^{(m)} = 0$. The τ-function is introduced through $\omega = \mathrm{d} \log \tau$.

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